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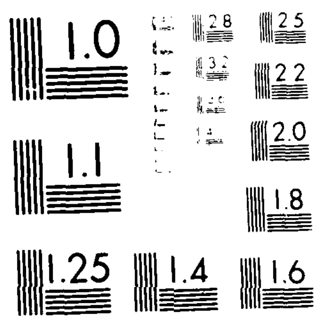
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TESTING WHETHER NEW IS BETTER THAN USED OF A SPECIFIED AGE[1]

by

Myles Hollander, Dong Ho Park, and Frank Proschan

Florida State University, Temple University
and Florida State University

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ITEM #20, CONTINUED:

We introduce a "new better than used at t_0" (NBU-t_0) class of life distributions, where the survival probability at age 0 is greater than or equal to the conditional survival probability at specified age $t_0 > 0$. The dual class of "new worse than used at t_0" (NWU-t_0) life distributions is defined by reversing the direction of inequality. We present preservation and non-preservation properties of the two classes under various reliability operations. We then develop a test of the null hypothesis that a new item has stochastically the same residual lifelength as does a used item of age t_0, versus the alternative hypothesis that a new item has stochastically greater residual lifelength than does a used item of age t_0.

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ABSTRACT

We introduce a "new better than used at t_0" (NBU-t_0) class of life distributions, where the survival probability at age 0 is greater than or equal to the conditional survival probability at specified age $t_0 > 0$. The dual class of "new worse than used at t_0" (NWU-t_0) life distributions is defined by reversing the direction of inequality. We present preservation and non-preservation properties of the two classes under various reliability operations. We then develop a test of the null hypothesis that a new item has stochastically the same residual lifelength as does a used item of age t_0, versus the alternative hypothesis that a new item has stochastically greater residual lifelength than does a used item of age t_0.

KEY WORDS: New Better than Used, New Better than Used of Age t_0, Failure Rate, Preservation Properties, Asymptotic Relative Efficiency.

1. INTRODUCTION

We introduce a "new better than used at t_0" (NBU-t_0) class of life distributions, where the survival probability at age 0 is greater than or equal to the conditional survival probability at specified age $t_0 > 0$. We also introduce the dual class of "new worse than used at t_0" (NWU-t_0) life dis-

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tributions obtained by reversing the direction of inequality. We then develop a test of the null hypothesis that a new item has stochastically the same residual lifelength as does a used item of age t_0, versus the alternative hypothesis that a new item has stochastically greater (smaller) residual lifelength than does a used item of age t_0.

Examples of situations where it is reasonable to use the NBU-t_0 (NWU-t_0) test are:

(i) From experience, cancer specialists believe that a patient newly diagnosed as having a certain type of cancer has a distinctly smaller chance of survival than does a patient who has survived 5 years ($=t_0$) following initial diagnosis. (In fact, such survivors are often designated as "cured".) The cancer specialists may wish to test their beliefs.

(ii) The Federal Aviation Administration requires an extensive overhaul of a commercial airplane engine after t_0 hours of flight. The airlines claim that this overhaul is unnecessary at best, and possibly is even harmful to the aircraft. To verify their claim, the airlines test from operational data whether an airplane engine after t_0 hours of flight is stochastically as good as a new engine.

(iii) A manufacturer believes that a certain component exhibits "infant mortality", for example, has a decreasing failure rate over an interval $[0, t_0]$. This belief stems from experience accumulated for similar components. He wishes to determine whether a used component of age t_0 has stochastically greater residual lifelength than does a new component. If so, he will test over the interval $[0, t_0]$ a certain percentage of his output and then sell the surviving components of age t_0 at higher prices to purchasers who must have high reliability components (e.g., a spacecraft assembler). He wishes to test such a hypothesis to reject or accept his a priori belief.

In Section 2 we present definitions and some preservation (and non-preservation) properties of the NBU-t_0 and NWU-t_0 classes under various reliability operations. In Section 3 we introduce an NBU-t_0 test based on a random sample from the underlying life distribution. The NBU-t_0 test is asymptotically distribution-free and is consistent against NBU-t_0 alternatives. In Section 4 we present Pitman asymptotic relative efficiency results for the NBU-t_0 test. We take the NBU test proposed by Hollander and Proschan (1972) [HP(1972)] as a competitor since other tests for the NBU-t_0 alternatives have not yet been proposed.

2. THE NBU-t_0 CLASS

Definition 2.1. Let $t_0 > 0$. A life distribution F is new better than used at t_0 (NBU-t_0) if

$$\overline{F}(x + t_0) \leq \overline{F}(x)\ \overline{F}(t_0) \quad \text{for all} \quad x \geq 0, \tag{2.1}$$

where $\overline{F} = 1\text{-}F$ denotes the survival function. The dual notion of new worse than used at t_0 (NWU-t_0) is defined analogously by reversing the first inequality in (2.1).

We define the following classes of life distributions.

$$C_0 = \{F\colon\ \overline{F}(x + t_0) = \overline{F}(x)\ \overline{F}(t_0) \quad \text{for all} \quad x \geq 0\}, \tag{2.2}$$

and

$$C_A = \{\overline{F}\colon\ \overline{F}(x + t_0) \leq \overline{F}(x)\ \overline{F}(t_0) \quad \text{for all} \quad x \geq 0 \\ \text{and inequality holds for some} \quad x \geq 0\}. \tag{2.3}$$

Then C_0 is the class of boundary members of the NBU-t_0 and NWU-t_0 classes. Using Theorem 2 of Marsaglia and Tubilla (1975), we may easily verify that the following distributions F_1, F_2, and F_3 are the only distributions in C_0:

(a) $\overline{F}_1(x) = \exp(-\lambda x)$, $\lambda > 0$, $x \geq 0$.

(b) $\overline{F}_2(x)$ is a survival function for which $\overline{F}_2(0) = 1$ and $\overline{F}_2(t_0) = 0$.

(c) $\overline{F}_3(x) = \overline{G}(x)$ for $0 \leq x < t_0$, $= \overline{G}^j(t_0)\,\overline{G}(x - jt_0)$ for $jt_0 \leq x < (j + 1)t_0$, $j = 0,1,2, \ldots$, where $\overline{G}$ is a survival function defined for $x \geq 0$. Note that if G has a density function on $[0, t_0]$, then the failure rate of F_3 is periodic with period t_0.

The NBU-t_0 is related to, but contains and is much larger than, the "new better than used" class defined below.

Definition 2.2. A life distribution F is new better than used (NBU) if $\overline{F}(x + t) \leq \overline{F}(x)\,\overline{F}(t)$ for all $x, t \geq 0$. F is new worse than used (NWU) if $\overline{F}(x + t) \geq \overline{F}(x)\,\overline{F}(t)$ for all $x, t \geq 0$.

Thus the NBU property states that a used item of *any* age has stochastically smaller residual lifelength than does a new item, whereas the NBU-t_0 property states that a used item of age t_0 has stochastically smaller residual lifelength than does a new item. The boundary members of the NBU and NWU classes are the exponential distributions.

Let C^* be the class of life distributions which are not NBU but are NBU-t_0. Theorem 2.3 below gives a method of constructing some distribution functions in C^*.

Given a survival function $\overline{H}$, let $\overline{H}_t(x) \equiv \overline{H}(t + x)/\overline{H}(t)$, the conditional survival function. Recall that for $x \geq 0$,

$$\overline{H}(x) = e^{-\int_0^x r_H(u)\,du} \tag{2.4a}$$

$$\overline{H}_t(x) = e^{-\int_t^{t+x} r_H(u)\,du} \tag{2.4b}$$

when H has failure rate function r_H.

Theorem 2.3. Let G be NBU with failure rate function $r_G(x) > 0$ for $0 \le x < \infty$. Let F have failure rate function r_F satisfying:

$$r_F(x) \le r_G(x) \quad \text{for} \quad 0 \le x \le t_0, \tag{2.5a}$$

$$r_F(x) = r_G(x) \quad \text{for} \quad t_0 \le x < \infty, \tag{2.5b}$$

and

$$r_F(x) \text{ is strictly decreasing for } 0 \le t \le t_1, \text{ where } 0 < t_1 < t_0. \tag{2.5c}$$

Then F is NBU-t_0 but not NBU.

Proof. F is not NBU by (2.5c).

To show that F is NBU-t_0, note that for $x \ge 0$,

$\overline{F}_{t_0}(x) = \overline{G}_{t_0}(x)$ by (2.4b) and (2.5b),

$\overline{G}_{t_0}(x) \le \overline{G}(x)$ since G is NBU,

and

$\overline{G}(x) \le \overline{F}(x)$ by (2.5a) and (2.5b).

Combining, we conclude that for $x \ge 0$,

$\overline{F}_{t_0}(x) \le \overline{F}(x)$;

i.e., F is NBU-t_0.||

Example 2.4. As an example of Theorem 2.3, let $r_G(x) = 1$ for $0 \le x < \infty$, and let $r_F(x) = 1 - (\theta/t_0)x$ for $0 \le x < t_0$ and $0 < \theta \le 1$, and $r_F(x) = 1$ for $t_0 \le x < \infty$. We do not let θ exceed 1 since we want to ensure that $r_F(x)$ remains positive as $x \to t_0$. Then r_F satisfies (2.5a), (2.5b), (2.5c) and thus F is in C^*. Using (2.4a) we can write $\overline{F}$ in the form given by (4.1). (In (4.1) we have extended the range of θ to include $\theta = 0$. When $\theta = 0$, the F of (4.1) is exponential and thus is both NBU and NBU-t_0.) We will utilize this $\overline{F}$ in the efficiency study of Section 4.

The remainder of Section 2 considers preservation properties of the NBU-t_0 and NWU-t_0 classes under various reliability operations.

Example 2.5. The NBU-t_0 class is not preserved under convolution: Let F be the distribution which places mass $\frac{1}{2} - \varepsilon$ at the point Δ_1, mass $\frac{1}{2} - \varepsilon$ at the point $\frac{5}{8} - \Delta_2$, and mass 2ε at the point $1 + \frac{1}{2}\Delta_1$, where Δ_1, Δ_2, and ε are "small" positive numbers. Let $t_0 = 1$. Then it is obvious that F is NBU-t_0 since a new item with life distribution F survives at least until time Δ_1 with probability 1, whereas an item of age t_0 has residual life $\frac{1}{2}\Delta_1$ with probability 1.

Next consider $\overline{F^{(2)}}(t_0 + x) = P[X_1 + X_2 > t_0 + x]$, where X_1, X_2 are i.i.d. $\sim F$, $t_0 = 1$ (as above), and $x = \frac{1}{4}$. Then $\overline{F^{(2)}}\,(\frac{5}{4}) = (\frac{1}{2} + \varepsilon)^2$, since $X_1 + X_2 > \frac{5}{4}$ if and only if $X_1 > \frac{5}{8} - \Delta_2$ *and* $X_2 \geq \frac{5}{8} - \Delta_2$.

Similarly, $\overline{F^{(2)}}(t_0) = \overline{F^{(2)}}(1) = P[X_1 + X_2 > 1] = (\frac{1}{2} + \varepsilon)^2 + 2(2\varepsilon)$ since $X_1 + X_2 > 1$ if and only if

(a) $X_1 > \frac{5}{8} - \Delta_2$ *and* $X_2 > \frac{5}{8} - \Delta_2$, or

(b) $X_1 = 1 + \frac{1}{2}\Delta_1$ *and* X_2 is any value, or

(c) $X_2 = 1 + \frac{1}{2}\Delta_2$ *and* X_1 is any value.

Finally, $\overline{F^{(2)}}(\frac{1}{4}) = 1 - (\frac{1}{2} - \varepsilon)^2$, since $X_1 + X_2 > \frac{1}{4}$ except when $X_1 = \Delta_1$ *and* $X_2 = \Delta_1$.

It follows that for $t_0 = 1$ and $x = \frac{1}{4}$, we have

$$\overline{F^{(2)}}(t_0 + x) - \overline{F^{(2)}}(t_0)\,\overline{F^{(2)}}(x) = (\tfrac{1}{2} + \varepsilon)^2 - [\{(\tfrac{1}{2} + \varepsilon)^2 + 2(2\varepsilon)\}\,\{1 - (\tfrac{1}{2} - \varepsilon)^2\}]$$

$$= \tfrac{1}{4} - (\tfrac{1}{4} \cdot \tfrac{3}{4}) + 0(\varepsilon)$$

which is greater than 0 for sufficiently small ε.

Thus $F^{(2)}$ is *not* NBU-t_0.||

Theorem 2.6. The NBU-t_0 class is preserved under the formation of coherent systems.

The proof is exactly analogous to that of the corresponding result for the NBU class. In the proof of Theorem 5.1 of Barlow and Proschan (1981), pp. 182-183, simply replace s by x and t by t_0.

Example 2.7. The NBU-t_0 class is not preserved under mixtures. The following example shows that a mixture of NBU-t_0 distributions need not be NBU-t_0. Let $\overline{F}_\alpha(x) = e^{-\alpha x}$ and $\overline{G}(x) = \int_0^\infty \overline{F}_\alpha(x)e^{-\alpha}d\alpha = (x + 1)^{-1}$. Then the density function is $g(x) = (x + 1)^{-2}$ and the failure rate function is $r_g(x) = (x + 1)^{-1}$, which is strictly decreasing in $x \geq 0$. Thus G is not NBU-t_0.

Example 2.8. The NWU-t_0 class is not preserved under convolution. The exponential distribution $F(x) = 1-e^{-x}$ is NWU-t_0. The convolution $F^{(2)}$ of F with itself is the gamma distribution of order 2: $F^{(2)}(x) = 1-(1 + x)e^{-x}$, with strictly increasing failure rate. Thus $F^{(2)}$ is not NWU-t_0.

Example 2.9. The NWU-t_0 class is not preserved under the formation of coherent systems.

This may be shown using the same example as is used for the analogous result for NWU systems in Barlow and Proschan (1981), p. 183.

Theorem 2.10. The NWU-t_0 class is (a) preserved under mixtures of distributions no two of which cross and (b) not preserved under arbitrary mixtures.

Proof (a) In the proof of the corresponding result for the NWU class (Barlow and Proschan, 1981, Theorem 5.7, p. 186), substitute x for s and t_0 for t.

(b) The example of 5.9, Barlow and Proschan (1981), p. 187, may be used.||

The preservation and non-preservation results for the NBU-t_0 and NWU-t_0 classes are summarized in Table 1 below.

Table 1. Preservation of NBU-t_0 (NWU-t_0) Under Reliability Operations.

	Convolutions	Formation of Coherent Systems	Arbitrary Mixtures	Mixtures of Noncrossing Distributions
NBU-t_0	Not Preserved	Preserved	Not Preserved	Not Preserved
NWU-t_0	Not Preserved	Not Preserved	Not Preserved	Preserved

3. THE NBU-t_0 TEST

We consider the problem of testing

$$H_0:\ F \text{ is in } C_0 \tag{3.1}$$

versus

$$H_A:\ F \text{ is in } C_A, \tag{3.2}$$

on the basis of a random sample $X_1, \ldots, X_n$ from a continuous distribution F. The classes C_0 and C_A are defined by (2.2) and (2.3), respectively. H_0 asserts that a new item is as good as a used item of age t_0, whereas H_A states that a new item has stochastically greater residual life than does a used item of age t_0. In what follows, " $\equiv$ " means "equals by definition".

Our test statistic is motivated by consideration of the parameter

$$\nu(F) \equiv \int_0^\infty \{\bar{F}(x+t_0) - \bar{F}(x)\,\bar{F}(t_0)\}\,dF(x) = \int_0^\infty \bar{F}(x+t_0)\,dF(x) - \tfrac{1}{2}\,\bar{F}(t_0) \equiv T_1(F) - T_2(F).$$

Under H_0, $T_1(F) = T_2(F)$ and thus $\nu(F) = 0$. Under H_A, $T_1(F) \le T_2(F)$ and thus $\nu(F) \le 0$. In fact, $\nu(F)$ is strictly less than 0 under H_A since F is continuous. It is reasonable to replace F by the empirical distribution F_n and reject H_0 in favor of H_A if $\nu(F_n)$ is too small. Roughly speaking, the more negative $\nu(F_n)$, the greater is the evidence in favor of H_A. Instead of using $\nu(F_n)$, we use the asymptotically equivalent U-statistic T defined by (3.3).

Let $h_1(X_1, X_2) = \frac{1}{2}\{\psi(X_1, X_2 + t_0) + \psi(X_2, X_1 + t_0)\}$ and $h_2(X_1) = \frac{1}{2}\,\psi(X_1, t_0)$ be the kernels of degree 2 and 1 corresponding to $T_1(F)$ and $T_2(F)$ respectively, where $\psi(a,b) = 1$ if $a > b$, $= 0$ if $a \le b$. Let

$$T = \{n(n-1)\}^{-1} \textstyle\sum' \psi(X_{\alpha_1}, X_{\alpha_2} + t_0) - (2n)^{-1} \displaystyle\sum_{i=1}^{n} \psi(X_i, t_0), \tag{3.3}$$

where $\sum'$ is the sum taken over all $n(n-1)$ sets of two integers (α_1, α_2) such that $1 \le \alpha_i \le n$, $i = 1, 2$, and $\alpha_1 \ne \alpha_2$. Let $\xi_1^{(1)} = E\{h_1(X_1, X_2)h_1(X_1, X_3)\} - \{T_1(F)\}^2$, $\xi_2^{(1)} = E\{h_1^2(X_1, X_2)\} - \{T_1(F)\}^2$, $\xi_1^{(2)} = E\{h_2^2(X_1)\} - \{T_2(F)\}^2$, and $\xi^{(1,2)} = E\{h_1(X_1, X_2) - T_1(F)\}\,\{h_2(X_1) - T_2(F)\}$.

Then

$$\mathrm{var}(T) = \binom{n}{2}^{-1} \sum_{k=1}^{2} \binom{2}{k}\binom{n-2}{2-k}\,\xi_k^{(1)} + n^{-1}\,\xi_1^{(2)} - (4/n)\,\xi^{(1,2)}. \tag{3.4}$$

and

$$\sigma^2 \equiv \lim_{n\to\infty} n.\mathrm{var}(T) = 4\xi_1^{(1)} + \xi_1^{(2)} - 4\xi^{(1,2)}.$$

Therefore from Hoeffding's (1948) U-statistic theory we have that if F is such that $\sigma^2 > 0$, then the limiting distribution of $n^{1/2}\{T - \nu(F)\}$ is normal with mean 0 and variance σ^2.

Straightforward calculations yield the following null hypothesis values: $\nu(F) = 0$, $4\xi_1^{(1)} = (1/3)\bar{F}(t_0) - (2/3)\bar{F}^2(t_0) + (1/3)\bar{F}^3(t_0)$, $4\xi_2^{(1)} = \bar{F}(t_0) - \bar{F}^2(t_0)$, $4\xi_1{}^{(2)} = \bar{F}(t_0) - \bar{F}^2(t_0)$, $4\xi^{(1,2)} = (1/2)\bar{F}(t_0) - \bar{F}^2(t_0) + (1/2)\bar{F}^3(t_0)$. Then from (3.4),

$$\mathrm{var}_0(T) = (n+1)\ \{n(n-1)\}^{-1}\ \{(1/12)\bar{F}(t_0) + (1/12)\bar{F}^2(t_0) - (1/6)\bar{F}^3(t_0)\},$$

and σ^2 reduces to σ_0^2, where

$$\sigma_0^2 = (1/12)\bar{F}(t_0) + (1/12)\bar{F}^2(t_0) - (1/6)\bar{F}^3(t_0). \tag{3.5}$$

The null mean of $n^{1/2}T$ is 0, independent of the underlying unspecified distribution F. However, the null variance and the null asymptotic variance of $n^{1/2}T$ depend on F through $\bar{F}(t_0)$ and thus σ_0^2 must be estimated from the data. Let

$$\hat{\sigma}_0^2 = (1/12)\bar{F}_n(t_0) + (1/12)\bar{F}_n^2(t_0) - (1/6)\bar{F}_n^3(t_0),$$

where $\bar{F}_n(t_0) = n^{-1} \cdot$ (number of X's $> t_0$). Then $\hat{\sigma}_0^2$ is a consistent estimator of σ_0^2 and, from the asymptotic normality of T and Slutsky's theorem, we thus have that under H_0,$n^{1/2}T\hat{\sigma}_0{}^{-1}$ is asymptotically $N(0,1)$. Thus our approximate α-level test of H_0 versus H_A (referred to as the NBU-t_0 test) rejects H_0 in favor of H_A if $n^{1/2}T\hat{\sigma}_0{}^{-1} \le -z_\alpha$, where z_α is the upper α-percentile point of a N(0,1) distribution. Analogously, the approximate α-level test of H_0 versus the alternative that a new item has stochastically less residual lifelength than does a used item of age t_0 rejects H_0 if $n^{1/2}T\hat{\sigma}_0{}^{-1} \ge z_\alpha$. This is referred to as the NWU-t_0 test.

We now show that if F is continuous, then the NBU-t_0 test is consistent against C_A. It suffices to show that $\nu(F)$ is strictly negative for $F \in C_A$. Let $D(x, t_0) = \bar{F}(x + t_0) - \bar{F}(x)\bar{F}(t_0)$. Then by assumption, $D(x, t_0) \le 0$ for all $x \ge 0$ and $D(x,t_0) < 0$ for some $x \ge 0$. Let x_0 be a point such that $D(x_0, t_0) < 0$ and let $x' = \sup\{x\colon\ x \ge x_0 \text{ and } \bar{F}(x) = \bar{F}(x_0)\}$. Then

$$D(x', t_0) = \bar{F}(x' + t_0) - \bar{F}(x')\bar{F}(t_0) \le \bar{F}(x_0 + t_0) - \bar{F}(x')\bar{F}(t_0)$$

$$= \bar{F}(x_0 + t_0) - \bar{F}(x_0)\bar{F}(t_0) = D(x_0, t_0) < 0.$$

Since F is continuous, D is also continuous. Therefore there exists a $\delta > 0$ such that $D(x' + \delta, t_0) < 0$. Also $F(x' + \delta) - F(x') > 0$, since x' is a point of increase of F. Thus $\nu(F) = \int_0^\infty D(x, t_0)dF(x) < 0$.

4. ASYMPTOTIC RELATIVE EFFICIENCY

To our knowledge, no other tests have been proposed for testing H_0 (3.1) versus H_A (3.2). Thus in this section we compare our NBU-t_0 test with the HP (1972) test of H_0': F is exponential, versus H_A': F is NBU and not exponential. The HP (1972) test statistic is
$J = 2\,\{n(n-1)\ (n-2)\}^{-1} \sum'' \psi(X_{\alpha_1}, X_{\alpha_2} + X_{\alpha_3})$, where $\sum''$ is the sum taken over all $n(n-1)(n-2)/2$ triplets $(\alpha_1, \alpha_2, \alpha_3)$ of three integers such that $1 \le \alpha_i \le n$, $i = 1,2,3$, $\alpha_1 \ne \alpha_2$, $\alpha_1 \ne \alpha_3$, and $\alpha_2 < \alpha_3$. The HP (1972) NBU test rejects H_0' in favor of H_A' for small values of J. (For large values of J, H_0' is rejected in favor of NWU alternatives.)

We evaluate the Pitman asymptotic relative efficiency of the HP (1972) NBU test with respect to our NBU-t_0 test for the following three distributions:

(1) The linear failure rate distribution given by

$$\bar{F}_1(x;\ \theta) = \exp[-\{x + (\theta/2)x^2\}],\ \theta \geq 0,\ x \geq 0.$$

(2) The Makeham distribution given by

$$\bar{F}_2(x;\ \theta) = \exp[-\{x + \theta(x + \exp(-x)-1)\}],\ \theta \geq 0,\ x \geq 0.$$

(3) The C* distribution (see Example 2.1) given by

$$\begin{aligned}\bar{F}_3(x;\ \theta) &= \exp[-\{x - \theta(2t_0)^{-1}x^2\}],\ 0 \leq \theta \leq 1,\ 0 \leq x < t_0 \\ &= \exp[-\{x - \theta(2)^{-1}t_0\}],\ 0 \leq \theta \leq 1,\ x \geq t_0.\end{aligned} \tag{4.1}$$

For $\theta = 0$, F_1, F_2, and F_3 reduce to the exponential distribution and thus satisfy H_0 and H_0'. For $\theta > 0$, F_1 and F_2 are NBU. For $0 < \theta \leq 1$, as shown in Section 2, F_3 is in C_A but is not NBU. Using the results of HP (1972) and the null asymptotic distribution of T, we have for $i = 1, 2, 3$,

$$\begin{aligned}\mathrm{ARE}_{F_i}(J,T) &= \lim_{n\to\infty}\{\mathrm{var}_0(T)/\mathrm{var}_0(J)\}] \cdot \{\Delta_i'(0)/\nu_i'(0)\}^2 \\ &= [\{(1/12)e^{-t_0} + (1/12)e^{-2t_0} - (1/6)e^{-3t_0}\}/(5/432)] \\ &\quad \cdot \{\Delta_i'(0)/\nu_i'(0)\}^2 \\ &= (432/60)(e^{-t_0} + e^{-2t_0} - 2e^{-3t_0}) \cdot \{\Delta_i'(0)/\nu_i'(0)\}^2\end{aligned} \tag{4.2}$$

where $\Delta_i(\theta) = \iint \bar{F}_i(x + y;\ \theta)dF_i(x;\ \theta)dF_i(y;\ \theta)$ and $\nu_i(\theta) = \{\int \bar{F}_i(x + t_0;\ \theta)dF_i(x;\ \theta)\} - (2)^{-1}\bar{F}_i(t_0)$ are the asymptotic means of J and T for the alternatives $F_i(x;\ \theta)$ and $\Delta_i'(0)[\nu_i'(0)]$ is the derivative of $\Delta_i(\theta)[\nu_i(\theta)]$ with respect to θ evaluated at $\theta = 0$. Direct calculations yield

$$\Delta_1'(0) = -1/16,\ \nu_1'(0) = -4^{-1}t_0e^{-t_0},$$

$$\Delta_2'(0) = -1/36,\ \nu_2'(0) = 6^{-1}e^{-t_0}(e^{-t_0}-1),$$

$$\Delta_3'(0) = (1-e^{-2t_0}-2t_0e^{-2t_0}-4t_0^2e^{-2t_0})/16t_0,$$

$$\nu_3'(0) = \{2t_0e^{-3t_0}-e^{-t_0}+e^{-3t_0}\}/8t_0.$$

Then from (4.2) we obtain

$$\mathrm{ARE}_{F_1}(J,T) = 9\{e^{t_0} + 1 - 2e^{-t_0}\}/\{20t_0^2\},$$

$$\mathrm{ARE}_{F_2}(J,T) = \{e^{t_0} + 1 - 2e^{-t_0}\}/\{5(-1 + e^{-t_0})^2\},$$

$$\mathrm{ARE}_{F_3}(J,T) = (9/5)[1 - e^{-2t_0}\{1 + 2t_0 + 4t_0^2\}]^2$$

$$\cdot\ (e^{t_0} + 1 - 2e^{-t_0})/[1 - e^{-2t_0}\{1 + 2t_0\}]^2.$$

Table 2 gives the asymptotic relative efficiencies for $t_0 = .2(.2)2(1)5$ when the underlying distributions are F_1, F_2, and F_3. Since F_3 is not NBU but is NBU-t_0, we should not be surprised that T outperforms J for some (small) values of t_0. For large values of t_0, the efficiencies of the NBU-t_0 test with respect to the HP NBU test are low for F_1, F_2, and F_3. This is due in part to the fact that for values of t_0 that are large (e.g., larger than the mean or median), more than half of the observations do not affect T significantly - although they are of course not completely neglected.

Table 2. ARE of HP NBU Test with respect to NBU-t_0 Test.

$t_0\backslash F$:	F_1	F_2	F_3
.2	6.569	3.554	.579
.4	3.238	2.118	.526
.6	2.156	1.695	.253
.8	1.636	1.536	.032
1.0	1.342	1.493	.042
1.2	1.162	1.523	.400
1.4	1.047	1.607	1.186
1.6	.975	1.742	2.461
1.8	.933	1.929	4.281
2.0	.913	2.172	6.706
3.0	1.049	4.649	31.226
4.0	1.563	11.531	95.750
5.0	2.689	30.287	266.482

5. AN EXAMPLE

Bryson and Siddiqui (1969, p. 1483) give data corresponding to the survival periods (in days) of 43 patients from the date of diagnosis of chronic granulocytic leukemia. To apply our NBU-t_0 test, we choose $t_0 = 1825$ ($\approx$ 5 years) for illustrative purposes. We obtain $T = -.0449$, $\hat{\sigma}_0^2 = .0151$, and $(43)^{1/2}T\hat{\sigma}_0^{-1} = -2.40$ with a corresponding one-sided P value of .0083. Thus the NBU-1825 test strongly suggests that a newly diagnosed patient has stochastically greater residual life than does a patient after 5 years. (We note that Figure 2 of Bryson and Siddiqui indicates a decreasing trend in mean residual lifelength.)

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